

NILPOTENCE AND TORSION IN THE COHOMOLOGY OF THE STEENROD ALGEBRA

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ABSTRACT. In this paper we prove the existence of global nilpotence and global torsion bounds for the cohomology of any finite Hopf subalgebra of the Steenrod algebra for the prime 2. An explicit formula for computing such bounds is then obtained. This is used to compute bounds for $H^*(\mathcal{A}_n)$ for $n \leq 6$.

1. INTRODUCTION

Let R be any commutative algebra and $x \in R$. An element $y \in R$ is said to be x -torsion if $x^k y = 0$ for some k . R has x -torsion bound m if $x^m y = 0$ for any x -torsion $y \in R$. R has global torsion bound m if m is an x -torsion bound for all $x \in R$. R has global nilpotence bound m if $x^m = 0$ for all nilpotent $x \in R$. For any algebra R , $x \in R$ has nilpotence k if $x^k = 0$ and $x^{k-1} \neq 0$.

Let \mathcal{A} be the Steenrod algebra at prime 2. Let $\mathcal{A}(n_1, n_2, \dots)$ be the \mathbb{Z}_2 -submodule of \mathcal{A} generated by the Milnor basis elements $Sq(r_1, r_2, \dots)$ with $r_i < 2^{n_i}$ for all i . If $\exists k$ such that $n_i = 0$ for all $i > k$ then we will write $\mathcal{A}(n_1, \dots, n_k)$ for $\mathcal{A}(n_1, n_2, \dots)$. $\mathcal{A}(n_1, n_2, \dots)$ is a Hopf subalgebra of \mathcal{A} if and only if $n_v \geq \min\{n_{v-u}, n_u + u - v\}$ for all u, v with $v > u \geq 1$. Further, these are the only Hopf subalgebras of \mathcal{A} , [2, 3].

Let $\Gamma = \mathcal{A}(n_1, \dots, n_k)$ be any finite Hopf subalgebra of \mathcal{A} and $H^*(\Gamma) = \text{Ext}_\Gamma(\mathbb{Z}_2, \mathbb{Z}_2)$. Our main results are

Theorem I.1. $H^*(\Gamma)$ has a global nilpotence bound which is computable by Proposition 1–3 below.

Theorem I.2. Any global nilpotence bound for $H^*(\Gamma)$ is also a global torsion bound (and hence an x -torsion bound for any element $x \in H^*(\Gamma)$) and vice versa.

Thus the global nilpotence bounds in Theorem I.1 are also global torsion bounds for $H^*(\Gamma)$. None of the bounds given above are known to be the best possible, i.e. there may be smaller bounds than those given. The following propositions are those referred to in Theorem I.1.

Proposition 1 (W. H. Lin [6]). $H^*(\Gamma)$ is nilfree if and only if

- (a) $\Gamma = \mathcal{A}(0, \dots, 0, n_t, n_{t+1}, \dots)$ with $n_i \leq t \ \forall i$, or

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- (b) $\Gamma = \mathcal{A}(0, \dots, 0, 1, n_{t+1}, n_{t+2}, \dots)$ with $n_i \leq t+1 \ \forall i$ and $n_j = t+1$ for at least one j .

In this case, $H^*(\Gamma)$ trivially has global nilpotence bound 1.

Proposition 2. Suppose Γ is a finite Hopf subalgebra of \mathcal{A} ; and suppose

(a) there exists a finite family $\{\Lambda_j | 1 \leq j \leq n\}$ of Hopf subalgebras of Γ , such that for each j , Γ is obtained from Λ_j by the addition of one generator whose class is α_j in the same sense of Definition II.1;

(b) there exist integers $w_j \geq 0$ such that $\prod_{j=1}^n \alpha_j^{w_j} = 0$ in $H^*(\Gamma)$;

(c) $H^*(\Lambda_j)$ has global nilpotence bound m_j ;

then $H^*(\Gamma)$ has global nilpotence bound $\sum_{j=1}^n w_j m_j$.

In §II we will define the notion of a ‘ k, m -allowable’ Hopf subalgebra of $\Gamma \subset \mathcal{A}$ (Definition II.4). To such a Hopf algebra Γ we will associate Hopf subalgebras $\Gamma_{(k)}$ and $\Gamma_{(m)}$ (Definition II.3). We will also define a sense in which a Hopf algebra can be built up from a Hopf subalgebra ‘by the addition of one generator whose class is $h_{s,i}$ ’ (Definitions II.1 and II.2).

Proposition 3. For any Γ and any integers k, m such that Γ is k, m -allowable, Γ is obtained from $\Gamma_{(k)}$ (and also from $\Gamma_{(m)}$) by the addition of one generator whose class is $h_{n_k, k}$ (resp. $h_{n_m, m}$) and such that,

(a) $h_{n_k, k}^{2^{n_m - n_k - k}} h_{n_m, m} = 0$ if $n_m \geq n_k + k$,

(b) $h_{n_k, k} h_{n_m, m}^{2^{n_k + k - n_m}} = 0$ if $n_m < n_k + k$.

Further, if $H^*(\Gamma)$ is not nilfree then there exists at least one such pair of integers.

While there is always one pair of integers k, m so that Γ is k, m -allowable in Proposition 3 (unless $H^*(\Gamma)$ is nilfree), there are often several such pairs. The freedom in choosing between such pairs gives rise to several different strategies for computing global nilpotence bounds that trade simplicity of computation for decreased size of the bounds.

The following is by far the simplest formula, but yields very large bounds. We restrict ourselves to the case $\Gamma = \mathcal{A}_n = \mathcal{A}(n+1, n, \dots, 3, 2, 1)$.

Theorem I.3. $H^*(\mathcal{A}_n)$ has global nilpotence bound $2^{((n+1)(n+2)(n+3)/2)}$.

The next method of computation yields the best possible bounds that are attainable by the methods of Theorem I.1, but does so at the expense of requiring extensive calculations to compute most bounds.

For any k, m -allowable Γ (Definition II.4) define integers:

$$w_0 = \begin{cases} 2^{n_m - n_k - k} & \text{if } n_m \geq n_k + k \\ 1 & \text{if } n_m < n_k + k \end{cases} \quad \text{and} \quad w_1 = \begin{cases} 1 & \text{if } n_m \geq n_k + k \\ 2^{n_k + k - n_m} & \text{if } n_m < n_k + k. \end{cases}$$

Thus $h_{n_k, k}^{w_0} h_{n_m, m}^{w_1} = 0$ in Proposition 3.

Theorem I.4. For any finite Hopf subalgebra Γ ,

$$\text{Bound}(\Gamma) = \begin{cases} 1 & \text{if } H^*(\Gamma) \text{ is nilfree,} \\ \min_{\substack{k, m \\ \Gamma \text{ } k, m\text{-allowable}}} \{w_0 \cdot \text{Bound}(\Gamma_{(k)}) + w_1 \cdot \text{Bound}(\Gamma_{(m)})\} & \text{otherwise,} \end{cases}$$

is a global nilpotence bound for $H^*(\Gamma)$.

A comparison of bounds for $H^*(\mathcal{A}_n)$ for $n \leq 6$ given by these two methods is shown in Table 1. The bounds given by Theorem I.4 are known to be the best possible for $H^*(\mathcal{A}_0)$ and $H^*(\mathcal{A}_1)$. $H^*(\mathcal{A}_2)$ must have a global nilpotence bound ≥ 4 [11]. For $n \geq 3$ the global nilpotence bound for $H^*(\mathcal{A}_n)$ must be $\geq 2^{n+1}$ [5].

TABLE 1. Global nilpotence bounds for $H^*(\mathcal{A}_n)$
(scientific notation values rounded to 1 significant digit)

n	Bound from Theorem I.3	Bound from Theorem I.4
0	8	1
1	4096	3
2	1E9	9
3	1E18	96
4	4E31	3,294
5	4E50	267,282
6	7E75	60,896,016

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II. BACKGROUND AND NOTATION

Let R be a commutative algebra. An ideal $\mathcal{P} \subset R$ is *prime* if $\mathcal{P} \neq R$ and $xy \in \mathcal{P} \Rightarrow x \in \mathcal{P}$ or $y \in \mathcal{P}$ for any $x, y \in R$. An ideal \mathcal{Q} is *primary* if $\mathcal{Q} \neq R$ and $xy \in \mathcal{Q} \Rightarrow y \in \mathcal{Q}$ or $x^k \in \mathcal{Q}$ for some k . If \mathcal{I} is any ideal then the *radical* of \mathcal{I} is defined to be $r(\mathcal{I}) \equiv \{x \in R : x^k \in \mathcal{I} \text{ for some } k \in \mathbb{N}\}$. The radical of an ideal is an ideal. The radical of a primary ideal is prime. The radical of a prime ideal is the ideal itself. If R is Noetherian, every ideal contains a power of its radical. In a Noetherian algebra every ideal is a finite intersection of primary ideals. Such a decomposition $\mathcal{I} = \bigcap_{i=1}^n \mathcal{Q}_i$ is said to be *minimal* if the $\mathcal{P}_i = r(\mathcal{Q}_i)$ are all distinct and \mathcal{Q}_i does not contain $\bigcap_{j \neq i} \mathcal{Q}_j$ for any i or j . Every ideal in a Noetherian ring has a minimal decomposition and the \mathcal{P}_i are uniquely determined.

Let \mathcal{A}^* be the *dual* of \mathcal{A} . Milnor [9] showed that $\mathcal{A}^* \simeq \mathbb{Z}_2[\xi_1, \xi_2, \xi_3, \dots]$ with diagonal map $\psi : \mathcal{A}^* \rightarrow \mathcal{A}^* \otimes \mathcal{A}^*$ given by $\psi(\xi_m) = \sum_{a+b=m} \xi_a^{2^b} \otimes \xi_b$ on generators and extended to be an algebra homomorphism. For any subalgebra $\Gamma = \mathcal{A}(n_1, n_2, \dots)$ of \mathcal{A} we have

$$\Gamma^* \simeq \mathbb{Z}_2[\xi_1, \xi_2, \xi_3, \dots] / (\xi_1^{2^{n_1}}, \xi_2^{2^{n_2}}, \xi_3^{2^{n_3}}, \dots).$$

We will say that $\xi_i^j = 0$ in Γ^* if the class of ξ_i^j is zero in this quotient. For any graded \mathbb{Z}_2 -algebra Γ define the *cohomology* of Γ to be $H^*(\Gamma) \equiv \text{Ext}_{\Gamma^*}^*(\mathbb{Z}_2, \mathbb{Z}_2)$.

This may be computed via the *cobar resolution* which is described fully in [1].

We now give some technical definitions that will be required in this paper.

Definition II.1. Suppose $\Gamma = \mathcal{A}(n_1, n_2, \dots)$ is a finite Hopf subalgebra and Λ is a normal Hopf subalgebra of Γ such that $\Gamma/\Lambda = E[\bar{x}]$ where \bar{x} is the equivalence class of an indecomposable element $x \in \Gamma$. Let $\alpha \in H^{1,*}(\Gamma)$ be the class which is the image of the polynomial generator of $H^*(\Gamma/\Lambda)$ under the map induced in cohomology by the quotient map. In this situation we say that Γ is obtained from Λ by the addition of one generator α .

In this case, let $i : \Lambda \rightarrow \Gamma$ be the inclusion map and $i^* : H^{*,*}(\Gamma) \rightarrow H^{*,*}(\Lambda)$ the induced map.

Definition II.2. Suppose Γ is a Hopf subalgebra of \mathcal{A} for which $\xi_i^{2^s-1} \in \Gamma^*$ is primitive. In this case we define $h_{s,i}$ to be the corresponding class in $H^1(\Gamma)$.

Definition II.3. For any finite Hopf subalgebra $\Gamma = \mathcal{A}(n_1, n_2, \dots)$ define

$$\Gamma_{(j)} = \begin{cases} \mathcal{A}(n_1, \dots, n_{j-1}, n_j - 1, n_{j+1}, \dots) & \text{if } n_j \neq 0, \\ \Gamma & \text{if } n_j = 0. \end{cases}$$

Definition II.4. For any finite Hopf subalgebra $\Gamma = \mathcal{A}(n_1, n_2, \dots)$ and any integers k, m we say that Γ is k, m -allowable iff

- (a) $1 \leq k \leq m$;
- (b) $n_k > 0$ and $n_m > k$;
- (c) $n_k > n_i$ and $n_m > n_i + k, \forall i < k$;
- (d) $n_k > n_i + i - (k + m)$ and $n_m > n_i + i - m, \forall i < m$.

We will show in §IV that if Γ is k, m -allowable then $\Gamma_{(k)}$ and $\Gamma_{(m)}$ are Hopf subalgebras of Γ . A good reference for the background material in commutative algebra is [4]. Background on the Steenrod algebra and its subalgebras can be found in [8].

III. THE EXISTENCE OF GLOBAL BOUNDS

In this section we prove Theorem I.2. In order to do so we require the following result. Let $\Gamma = \mathcal{A}(n_1, \dots, n_k)$ be any finite Hopf subalgebra of \mathcal{A} .

Theorem III.1 (Wilkerson [12]). $H^*(\Gamma)$ is Noetherian.

Theorem III.2. Any Noetherian algebra R has bounded x -torsion $\forall x \in R$.

Proof. For any $x \in R$, define $\text{Ann}(x) = \{y \in R : yx = 0\}$. Consider the ascending chain of ideals:

$$\text{Ann}(x) \subseteq \text{Ann}(x^2) \subseteq \text{Ann}(x^3) \subseteq \dots \subseteq \text{Ann}(x^n) \subseteq \text{Ann}(x^{n+1}) \subseteq \dots$$

Since R is Noetherian, this ascending chain must become stationary. Thus there is m_x such that $\text{Ann}(x^k) = \text{Ann}(x^{m_x})$ for all $k \geq m_x$ and m_x is the least such integer. Hence, $x^k y = 0 \Rightarrow x^{m_x} y = 0$. \square

The following theorem was shown to the author by G. Stengle.

Theorem III.3. Every Noetherian algebra has a global torsion bound.

Proof. Let R be Noetherian and (0) be the zero ideal in R . Then we have a minimal primary decomposition for (0) , i.e., $(0) = \bigcap_{i=1}^n \mathcal{Q}_i$ where the \mathcal{Q}_i

are primary ideals and $n \in \mathbb{N}$. Define $\mathcal{P}_i = r(\mathcal{Q}_i)$ for each $i \leq n$. Let m_i be the smallest integer such that $\mathcal{P}_i^{m_i} \subset \mathcal{Q}_i$. Let $m = \max_i\{m_i\}$. Suppose $x^k y = 0$. Then $x^k y \in (0) = \bigcap_{i=1}^n \mathcal{Q}_i$ and so $x^k y \in \mathcal{Q}_i$ for each $i \leq n$. Since \mathcal{Q}_i is primary, $y \in \mathcal{Q}_i$ or $x^{kj} \in \mathcal{Q}_i$ for some j , which implies that $y \in \mathcal{Q}_i$ or $x \in \mathcal{P}_i$. Thus $y \in \mathcal{Q}_i$ or $x^m \in \mathcal{Q}_i$. So $x^m y \in \mathcal{Q}_i$ for any $i \leq n$. Hence $x^m y \in \bigcap_{i=1}^n \mathcal{Q}_i = (0)$. So $x^m y = 0$. \square

Theorem III.4. *Every Noetherian algebra R has a global nilpotence bound.*

Proof. This is an obvious corollary to the previous theorem (take $y = 1$). But it has a simpler proof. Since R is Noetherian the ideal (0) must contain some power of $r(0)$, which is just the set of nilpotents in R . \square

Theorem III.5. *In any Noetherian algebra the smallest global torsion bound and the smallest global nilpotence bound are equal.*

Proof. As in the proof of Theorem III.3, let R be Noetherian and (0) be the zero ideal in R . Then we have a minimal primary decomposition for (0) , i.e. $(0) = \bigcap_{i=1}^n \mathcal{Q}_i$ where the \mathcal{Q}_i are primary ideals and $n \in \mathbb{N}$. Define $\mathcal{P}_i = r(\mathcal{Q}_i)$ for each $i \leq n$. Let m_i be the smallest integer such that $\mathcal{P}_i^{m_i} \subset \mathcal{Q}_i$. Let $m = \max_i\{m_i\}$. We first show that m is the smallest global torsion bound for R . If $m = 1$ it is clearly minimal. If $m > 1$ then since $m_j = m$ for some j , we can choose $z \in \mathcal{P}_j$ so that $z^m \in \mathcal{Q}_j$ and $z^{m-1} \notin \mathcal{Q}_j$. Since our decomposition is minimal there is $w \in \bigcap_{i \neq j} \mathcal{Q}_i$ such that $w \notin \mathcal{Q}_j \subseteq \mathcal{P}_j$. Then $z^{m-1} w \notin \mathcal{Q}_j$ since $z^{m-1} \notin \mathcal{Q}_j$ and $w \notin \mathcal{P}_j$ and \mathcal{Q}_j is primary. So m is the smallest global torsion bound.

Now let b = the smallest global nilpotence bound. Clearly $m \geq b$ since $x^k = 0 \Rightarrow x^k \cdot 1 = 0$. It follows that if $m = 1$ then $m = b = 1$. If $m > 1$ then since $z^m w = 0$ we have $(zw)^m = (z^m w) w^{m-1} = 0 w = 0$. But $(zw)^{m-1} = (z^{m-1} w) w^{m-2}$ (where $w^0 = 1$). Further, $z^{m-1} w \notin \mathcal{Q}_j$ and $w^{m-2} \notin \mathcal{P}_j$ (because $r(\mathcal{P}_j) = \mathcal{P}_j$ and $w \notin \mathcal{P}_j$), their product $(zw)^{m-1} \notin \mathcal{Q}_j$ because \mathcal{Q}_j is primary. Thus $(zw)^{m-1} \notin \bigcap_{i=1}^n \mathcal{Q}_i = (0) \Rightarrow (zw)^{m-1} \neq 0$. So zw has nilpotence m . Thus $b \geq m$. So $m = b$. \square

Thus the question of finding a global torsion bound is equivalent to that of finding the global nilpotence bound. Hence we now have

Proof (of Theorem I.2). By Theorem III.1 $H^*(\Gamma)$ is Noetherian. Thus by Theorem III.3 it has a global torsion bound, and hence a smallest one. By Theorem III.5 this is equal to the smallest global nilpotence bound. The theorem follows. \square

Corollary III.1. *The bounds computed by Theorem I.1 are also global torsion bounds.*

IV. COMPUTING GLOBAL BOUNDS

In this section we give the proof of Theorem I.1. It suffices to prove Theorem I.4 since this satisfies the existence requirement of Theorem I.1. We begin by showing how Theorem I.4 follows from Propositions 1–3.

Proof (of Theorem I.4). For any finite Hopf subalgebra $\Gamma = \mathcal{A}(n_1, n_2, \dots)$ define $\sigma(\Gamma) = \sum_i n_i$. Notice that Γ finite $\Rightarrow \sigma(\Gamma) < \infty$. We proceed by

induction on $\sigma(\Gamma)$. If $\sigma(\Gamma) = 1$ then Γ is exterior and $H^*(\Gamma)$ is nilfree by Proposition II.1. Hence $\text{Bound}(\Gamma) = 1$ and this is trivially a global nilpotence/torsion bound for $H^*(\Gamma)$. So assume that $\text{Bound}(\Lambda)$ is a global nilpotence/torsion bound for the cohomology of any Λ with $\sigma(\Lambda) < \sigma(\Gamma)$. If Γ has nilfree cohomology, then $\text{Bound}(\Gamma) = 1$ and is still trivially a global nilpotence/torsion bound. If not, then by Proposition 3 there is at least one pair of integers k, m so that Γ is obtained from $\Gamma_{(k)}$ (resp. $\Gamma_{(m)}$) by the addition of one generator α_k (resp. α_m) so that $\alpha_k^{w_0} \alpha_m^{w_1} = 0$. By the induction hypothesis, $\text{Bound}(\Gamma_{(i)})$ is a global nilpotence/torsion bound for $H^*(\Gamma_{(i)})$ for any i such that $\Gamma_{(i)}$ is a proper subalgebra (i.e., $n_i \neq 0$), since then $\sigma(\Gamma_{(i)}) < \sigma(\Gamma)$. Thus by Proposition 2, $H^*(\Gamma)$ has global nilpotence/torsion bound $w_0 \cdot \text{Bound}(\Gamma_{(k)}) + w_1 \cdot \text{Bound}(\Gamma_{(m)})$. Since this will be true for any choice of the pair k, m such that Γ is k, m -allowable, we can take the minimum of all such values to get the best possible value using these methods. This is precisely what $\text{Bound}(\Gamma)$ is, completing the induction and hence the proof. \square

So we must now prove Propositions 1–3. Proposition 1 was proved by W. H. Lin [6].

In order to prove Proposition 2 we require the following result of Lin [6].

Theorem IV.1. *If Γ is obtained from Λ by the addition of one generator whose class is α and $i^* : H^{*,*}(\Gamma) \rightarrow H^{*,*}(\Lambda)$ is the map induced by the inclusion, then $\text{Ker}(i^*) = \text{ideal of } H^*(\Gamma) \text{ generated by } \alpha$.*

Using this result we can now prove Proposition 2, which is the key result used in this method of computing global nilpotence/torsion bounds.

Proof (of Proposition 2). Let $x \in H^*(\Gamma)$ be nilpotent. Let $i_j : \Lambda_j \rightarrow \Gamma$ be the inclusion map and let $M = \sum_{j=1}^n w_j m_j$. Then x nilpotent $\Rightarrow x^k = 0$ for some $k \Rightarrow i_j^*(x)^k = i_j^*(x^k) = 0$ for each $j = 1, \dots, n \Rightarrow i_j^*(x)$ is nilpotent for each $j \Rightarrow i_j^*(x)^{m_j} = 0$ for each $j \Rightarrow i_j^*(x^{m_j}) = 0$ for each $j \Rightarrow x^{m_j} \in \text{Ker } i_j^*$ for each $j \Rightarrow x^{m_j} = \alpha_j \cdot y_j$ for each j and some $y_j \in H^*(\Gamma)$ (by Theorem IV.1) $\Rightarrow x^M = \prod_{j=1}^n (x^{m_j})^{w_j} = (\prod_{j=1}^n \alpha_j^{w_j}) \cdot y = 0 \cdot y = 0$ for some $y \in \Gamma$. \square

It should be noted at this point that we will only require the cases $n = 1, 2$ for our purposes.

It now remains to prove Proposition 3. In doing so we imitate the methods of Lin [6]. We proceed by a series of lemmas. Throughout we will let $\Gamma = \mathcal{A}(n_1, n_2, \dots)$ be any finite Hopf subalgebra of \mathcal{A} .

Lemma IV.1. *$h_{s,t}$ is a well-defined nonzero element of $H^*(\Gamma)$ if $n_i + i - t + 1 \leq s \leq n_t$ for all $i < t$.*

Proof. Since $[\xi_t^{2^{s-1}}]$ cannot be a coboundary it is enough to show it is a cocycle. But $d[\xi_t^{2^{s-1}}] = \sum_{i < t} [\xi_i^{2^{t-i+s-1}} | \xi_{t-i}^{2^{s-1}}] = 0$ since $n_i \leq t - i + s - 1 \Rightarrow \xi_i^{2^{t-i+s-1}}$ is zero in $\Gamma^* = \mathcal{A}(n_1, n_2, \dots)^*$. \square

For any rational number q let $\llbracket q \rrbracket$ be the greatest integer less than or equal to q . There are Steenrod operations in cohomology [7].

Lemma IV.2. *There are Steenrod operations on $H^{*,*}(\Gamma)$ satisfying:*

1. $Sq^i : H^{s,t}(\Gamma) \rightarrow H^{s+i, 2t}(\Gamma)$,

2. $Sq^i(ab) = \sum_{j+k=i} Sq^j(a)Sq^k(b),$
3. $Sq^r Sq^s = \sum_{t=0}^{\lfloor r/2 \rfloor} \binom{s-t-1}{r-2t} Sq^{r+s-t} Sq^t,$
4. $Sq^0([\alpha_1 | \dots | \alpha_n]) = [\alpha_1^2 | \dots | \alpha_n^2],$
5. $Sq^s(x) = x^2$ if $x \in H^{s,*}(\Gamma),$
6. $Sq^s(x) = 0$ if $x \in H^{r,*}(\Gamma)$ and $r < s.$

Thus by property 4 we have $Sq^0(h_{s,t}) = h_{s+1,t}$ whenever these are defined.

Lemma IV.3. *Let x, y be nonzero elements of $H^{1,*}(\Gamma).$*

Define $Q_b = Sq^{2^{b-1}} Sq^{2^{b-2}} \dots Sq^4 Sq^2 Sq^1.$ Then

$$Q_b(xy) = (x^{2^b})((Sq^0)^b y) + ((Sq^0)^b x)(y^{2^b}).$$

Proof. This is proved in [6]. There is an easy proof using Lemma IV.2. \square

Lemma IV.4. *If Γ is k, m -allowable then*

- (a) $h_{n_k+k,m}$ and $h_{n_k,k}$ are well-defined nonzero elements of $H^1(\Gamma)$ if $n_m \geq n_k + k,$ and
- (b) $h_{n_m,m}$ and $h_{n_m-k,k}$ are well-defined nonzero elements of $H^1(\Gamma)$ if $n_m < n_k + k.$

Proof. Case a: $n_m \geq n_k + k.$ Then Γ k, m -allowable $\Rightarrow n_i + i - k < n_i < n_k$ for $i < k \Rightarrow h_{n_k,k} \neq 0 \in H^1(\Gamma)$ by Lemma IV.1, and Γ is k, m -allowable $\Rightarrow n_i + i - m < n_k + k \leq n_m$ for $i < m \Rightarrow h_{n_k+k,m} \neq 0 \in H^1(\Gamma)$ by Lemma IV.1.

Case b: $n_m < n_k + k.$ Then Γ is k, m -allowable $\Rightarrow n_i + i - k < n_i < n_m - k < n_k$ for $i < k \Rightarrow h_{n_m-k,k} \neq 0 \in H^1(\Gamma)$ by Lemma IV.1, and Γ is k, m -allowable $\Rightarrow n_i + i - m < n_m$ for $i < m \Rightarrow h_{n_m,m} \neq 0 \in H^1(\Gamma)$ by Lemma IV.1. \square

Lemma IV.5. *If Γ is k, m -allowable then*

- (a) $h_{n_k+k,m} h_{n_k,k} = 0$ in $H^*(\Gamma)$ if $n_m \geq n_k + k,$ and
- (b) $h_{n_m,m} h_{n_m-k,k} = 0$ in $H^*(\Gamma)$ if $n_m < n_k + k.$

Proof. Case a: $n_m \geq n_k + k.$ Then Γ is k, m -allowable $\Rightarrow n_i < n_k$ for $i < k \Rightarrow \xi_i^{2^{n_k-1}} = 0$ in Γ^* for $i < k,$ and Γ is k, m -allowable $\Rightarrow n_j < n_k + k + m - j$ for $j < m \Rightarrow \xi_j^{2^{n_k+k+m-j-1}} = 0$ in Γ^* for $j < m.$ Thus in the cobar resolution for $H^*(\Gamma):$

$$\begin{aligned} d[\xi_{k+m}^{2^{n_k-1}}] &= \sum_{i=1}^{k+m-1} [\xi_{k+m-i}^{2^{n_k+i-1}} | \xi_i^{2^{n_k-1}}] \\ &= \sum_{i < k} [\xi_{k+m-i}^{2^{n_k+i-1}} | \xi_i^{2^{n_k-1}}] + \sum_{j < m} [\xi_j^{2^{n_k+k+m-j-1}} | \xi_{k+m-j}^{2^{n_k-1}}] + [\xi_m^{2^{n_k+k-1}} | \xi_k^{2^{n_k-1}}] \\ &= [\xi_m^{2^{n_k+k-1}} | \xi_k^{2^{n_k-1}}]. \end{aligned}$$

Hence, $h_{n_k+k,m} h_{n_k,k} = 0$ in $H^*(\Gamma).$

Case b: $n_m < n_k + k.$ Then Γ is k, m -allowable $\Rightarrow n_i < n_m - k$ for $i < k \Rightarrow \xi_i^{2^{n_m-k-1}} = 0$ in Γ^* for $i < k,$ and Γ is k, m -allowable $\Rightarrow n_j < n_m + m - j$ for $j < m \Rightarrow \xi_j^{2^{n_m+m-j-1}} = 0$ in Γ^* for $j < m.$ Thus in the cobar resolution for

$H^*(\Gamma) :$

$$\begin{aligned} d[\xi_{k+m}^{2^{n_m-k-1}}] &= \sum_{i=1}^{k+m-1} [\xi_{k+m-i}^{2^{n_m+i-k-1}} | \xi_i^{2^{n_m-k-1}}] \\ &= \sum_{i < k} [\xi_{k+m-i}^{2^{n_m+i-k-1}} | \xi_i^{2^{n_m-k-1}}] \\ &\quad + \sum_{j < m} [\xi_j^{2^{n_k+m-j-1}} | \xi_{k+m-j}^{2^{n_m-k-1}}] + [\xi_m^{2^{n_m-1}} | \xi_k^{2^{n_m-k-1}}] \\ &= [\xi_m^{2^{n_m-1}} | \xi_k^{2^{n_m-k-1}}]. \end{aligned}$$

Hence, $h_{n_m, m} h_{n_m-k, k} = 0$ in $H^*(\Gamma)$. \square

Lemma IV.6. *If Γ is k, m -allowable then*

- (a) $h_{n_m, m} h_{n_k, k}^{2^{n_m-n_k-k}} = 0$ in $H^*(\Gamma)$ if $n_m \geq n_k + k$, and
- (b) $h_{n_m, m}^{2^{n_k+k-n_m}} h_{n_k, k} = 0$ in $H^*(\Gamma)$ if $n_m < n_k + k$.

Proof. Notice that by definition $h_{n_i+b, i} = 0$ in $H^*(\Gamma)$ for any i if $b > 0$.

Case a: $n_m \geq n_k + k$. If $n_m = n_k + k$ then we are done by Lemma IV.5. So assume $n_m > n_k + k$ and let $b = n_m - n_k - k$. Then by Lemma IV.3 and Lemma IV.4,

$$\begin{aligned} 0 &= Q_b(0) = Q_b(h_{n_k+k, m} h_{n_k, k}) \\ &= h_{n_k+k, m}^{2^b} h_{n_k+b, k} + h_{n_k+k+b, m} h_{n_k, k}^{2^b} = h_{n_m, m} h_{n_k, k}^{2^{n_m-n_k-k}}. \end{aligned}$$

Case b: $n_m < n_k + k$. Let $b = n_m - n_k - k$. Then by Lemma IV.3 and Lemma IV.4,

$$\begin{aligned} 0 &= Q_b(0) = Q_b(h_{n_m, m} h_{n_m-k, k}) \\ &= h_{n_m, m}^{2^b} h_{n_m-k+b, k} + h_{n_m+b, m} h_{n_m-k, k}^{2^b} = h_{n_m, m}^{2^{n_k+k-n_m}} h_{n_k, k}. \quad \square \end{aligned}$$

Lemma IV.7. *If $\Gamma_{(i)}$ is a proper Hopf subalgebra of Γ then Γ is obtained from $\Gamma_{(i)}$ by the addition one generator $h_{n_i, i}$.*

Proof. Since $\Gamma_{(i)}$ is a proper subalgebra $n_i > 0$. Let $\Lambda = \Gamma_{(i)}$ and $q = n_i - 1$. We must show Λ is normal in Γ , i.e., that $\Gamma \cdot \bar{\Lambda} = \bar{\Lambda} \cdot \Gamma$ where $\bar{\Lambda}$ is the augmentation ideal. To accomplish this we imitate the methods of Margolis [8]. It suffices to show that

$$\mathcal{E} = \mathbb{Z}_2 \text{ vector space span of } \{ Sq(r_1, \dots) \in \Gamma \mid Sq(r_1, \dots) \neq P_i^q \}$$

spans both $\Gamma \cdot \bar{\Lambda}$ and $\bar{\Lambda} \cdot \Gamma$. If x and y are Milnor basis elements then P_i^q is a summand of xy if and only if $x = P_j^{q+i-j}$ and $y = P_{i-j}^q$ for some j [8]. By hypothesis Λ is a Hopf subalgebra of Γ and so either $n_j \leq q+i-j$ or $n_{i-j} \leq q$ for any $j < i$. Thus P_j^{q+i-j} and P_{i-j}^q are not in Γ for any $j < i$. Thus for any Milnor elements $x, y \in \Gamma$, P_i^q is not a summand of xy or yx unless $x = 1$ or $y = 1$. so $\Gamma \cdot \bar{\Lambda} \subset \mathcal{E}$ and $\bar{\Lambda} \cdot \Gamma \subset \mathcal{E}$. To obtain the opposite inclusions we

note that as shown in [8], for any Milnor element $Sq(r_1, \dots) \in \mathcal{E}$,

$$\begin{aligned} Sq(r_1, \dots) &= Sq(r_1, \dots, r_j - 2^h, \dots) \cdot P_j^h \\ &\quad + \text{terms of lower excess than } Sq(r_1, \dots) \\ &= P_j^h \cdot Sq(r_1, \dots, r_j - 2^h, \dots) \\ &\quad + \text{terms of lower excess than } Sq(r_1, \dots). \end{aligned}$$

Notice that here that P_i^q is not one of the terms of lower excess (nor is it $Sq(r_1, \dots)$ since this is in \mathcal{E}) because if it were it would be a summand of the product $Sq(r_1, \dots, r_j - 2^h, \dots) \cdot P_j^h$ (resp. $P_j^h \cdot Sq(r_1, \dots, r_j - 2^h, \dots)$) which we have already shown to be impossible since neither factor is 1. Hence by induction on excess, $Sq(r_1, \dots) \in \mathcal{E} \Rightarrow Sq(r_1, \dots) \in \Gamma \cdot \bar{\Lambda}$ and $Sq(r_1, \dots) \in \bar{\Lambda} \cdot \Gamma$. So $\mathcal{E} \subset \bar{\Lambda} \cdot \Gamma$ and $\mathcal{E} \subset \Gamma \cdot \bar{\Lambda}$ and so Λ is normal in Γ . Thus the quotient $\Gamma/\Lambda = \Gamma/\Gamma \cdot \bar{\Lambda} = E[P_i^q]$ is thus well defined and so Γ is obtained from Λ by the addition of one generator whose cobar representative is $[\xi_i^{2^q}] = [\xi_i^{2^{n_i-1}}]$, i.e., $h_{n_i, i}$. \square

Lemma IV.8. *If Γ is k, m -allowable then $\Gamma_{(k)}$ and $\Gamma_{(m)}$ are Hopf subalgebras of Γ .*

Proof. Let

$$\begin{aligned} \Gamma &= \mathcal{A}(n_1, n_2, \dots), \\ \Gamma_{(k)} &= \mathcal{A}(p_1, p_2, \dots) = \mathcal{A}(n_1, n_2, \dots, n_k - 1, \dots), \end{aligned}$$

and

$$\Gamma_{(m)} = \mathcal{A}(q_1, q_2, \dots) = \mathcal{A}(n_1, n_2, \dots, n_m - 1, \dots).$$

Since Γ is a Hopf subalgebra of \mathcal{A} we have either $n_v \geq n_u + u - v$ or $n_v \geq n_{v-u}$ for all $1 \leq u < v$, and must show this property holds for both the p_i and the q_i .

For $\Gamma_{(k)}$, the lemma easily follows from the following facts. If $v < k$ then $p_i = n_i$ for all $i \leq v$. If $v = k$ then $p_k = n_k - 1 \geq n_u \geq n_u + (u - k)$ for all $u < v$ since Γ is k, m -allowable. Finally, if $v > k$ then $p_u \leq n_u$ for all $u < v$ and $p_v = n_v$. Similarly for $\Gamma_{(m)}$, the lemma easily follows from the following facts. If $v < m$ then $q_i = n_i$ for all $i \leq v$. If $v = m$ then $q_m = n_m - 1 \geq n_u + (u - m)$ for all $u < v$ since Γ is k, m -allowable. Finally, if $v > m$ then $q_u \leq n_u$ for all $u < v$ and $q_v = n_v$. \square

Using these results we can now prove Proposition 3.

Proof (of Proposition 3). Γ is k, m -allowable $\Rightarrow \Gamma_{(k)}$ (and $\Gamma_{(m)}$) are Hopf subalgebras of Γ (by Lemma IV.8) which Γ is obtained from by the addition of one generator whose class is $h_{n_k, k}$ (resp. $h_{n_m, m}$) (by Lemma IV.7). The relations (a) and (b) in the proposition are then true by Lemma IV.6. Thus it suffices to show that for any $\Gamma = \mathcal{A}(n_1, n_2, \dots)$ which is not of type (a) of Proposition 1, there is one pair of integers k, m such that Γ is k, m -allowable. Let $k = \min\{j: n_j \neq 0\}$ and let $m = \min\{j: n_j > k\}$ (which must exist since Γ is not of type (a) of Proposition 1). We now show that Γ is k, m -allowable (refer to Definition II.4). Conditions (a) and (b) follow trivially from the definition of k and m . By definition of $k, n_i = 0$ for $i < k$, so condition

(c) follows. Finally, by definition of m , $n_i \leq k$ for $i < m$ and $n_m > k$ so $n_m > k \geq n_i > n_i + i - m$ for $i < m$ and $n_k > 0 \geq n_i - k > n_i - k + i - m$ for $i < m$. \square

All that remains is to prove Theorem I.3.

Proof (of Theorem I.3). Let $n \in \mathbb{N}$ and $W = 2^{n+2}$. If we write $\mathcal{A}_n = \mathcal{A}(n_1, n_2, \dots, n_{n+1}) = \mathcal{A}(n+1, n, \dots, 2, 1)$ then $n_i = n+2-i$ for $1 \leq i \leq n+1$. Then $w_i \leq 2^{|n_m - n_k - k|} \leq 2^{n_k + k} \leq 2^{(n+2-k)+k} = 2^{n+2} = W$ for all w_i involved in the calculation of $\text{Bound}(\mathcal{A}_n)$ in Theorem I.4. For any finite Hopf subalgebra of \mathcal{A}_n , $\Gamma = \mathcal{A}(n_1, n_2, \dots)$, define $\text{Toobig}(\Gamma) = (2W)^{\sigma(\Gamma)} = 2^{(n+3)\sigma(\Gamma)}$ where $\sigma(\Gamma) = \sum_i n_i$. Assume towards induction that $\text{Toobig}(\Lambda) \geq \text{Bound}(\Lambda)$ for any Λ with $\sigma(\Lambda) < \sigma(\Gamma)$. Then $\text{Toobig}(\Gamma) \geq \text{Bound}(\Gamma)$ since

$$\begin{aligned} \text{Bound}(\Gamma) &\leq \min_{\beta(k, m, \Gamma)=1} \{w_0 \cdot \text{Bound}(\Gamma_{(k)}) + w_1 \cdot \text{Bound}(\Gamma_{(m)})\} \\ &\leq 2 \cdot W \cdot \text{Toobig}(\Gamma_{(i)}), \quad \forall i \text{ with } \Gamma_{(i)} \neq \Gamma \\ &\leq 2 \cdot W \cdot (2^{(n+3)\sigma(\Gamma_{(i)})}) = 2^{(n+3)(\sigma(\Gamma_{(i)})+1)} \\ &= 2^{(n+3)\sigma(\Gamma)} = \text{Toobig}(\Gamma). \end{aligned}$$

Taking $\Gamma = \mathcal{A}_n$, we have $\sigma(\Gamma) = (n+1)(n+2)/2$ and so $\text{Toobig}(\mathcal{A}_n) = 2^{((n+1)(n+2)(n+3)/2)}$ is a global nilpotence bound for $H^*(\mathcal{A}_n)$. \square

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